Some cases of Vojta's Conjecture on integral points over function fields

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Abstract. In the present paper we solve in particular the function field version of a special case of Vojta's conjecture for integral points, namely for the variety obtained by removing a conic and two lines from the projective plane. This will follow from a bound for the degree of a curve on such a surface in terms of its Euler characteristic.

This case is special, but significant, because it lies "at the boundary", in the sense that it represents the simplest case of the conjecture which is still open. Also, it was studied in the context of Nevanlinna Theory by M. Green already in the seventies.

Our general results immediately imply the degeneracy of solutions of Fermat's type equations $z^d = P(x^m, y^n)$ for all $d \geq 2$ and large enough m, n, also in the case of non-constant coefficients. Such equations fall apparently out of all known treatments.

The methods used here refer to derivations, as is usual in function fields, but contain fundamental new points. One of the tools concerns an estimation for the $\gcd(1-u, 1-v)$ for S-units u, v; this had been developed also in the arithmetic case, but for function fields we may obtain a much more uniform quantitative version.

In an Appendix we shall finally point out some other implications of the methods to the problem of torsion-points on curves and related known questions.

§1. Introduction and main results.

A celebrated conjecture in diophantine geometry, first proposed by Vojta, reads as follows: *let X be a smooth affine variety defined over a number field k, $\tilde{X}$ be a projective variety containing X as an open subset, $D = \tilde{X} \setminus X$ the divisor at infinity and K a canonical divisor of $\tilde{X}$. Suppose that D is a normal crossing divisor. Then if $D + K$ has maximal Kodaira dimension, for every ring of S-integers $\mathcal{O}_S \subset k$, the set of S-integral points $X(\mathcal{O}_S)$ is not Zariski-dense.*

It is known that the Kodaira dimension of $D + K$ is in fact independent of the smooth compactification $\tilde{X}$ of X, provided that D has normal crossings (see for instance [KMK]). Following a frequent notation, we will call log Kodaira dimension of X the Kodaira dimension of $D + K$.

A complex analytic analogue of Vojta's Conjecture asks for the degeneracy of entire curves on affine varieties with maximal log Kodaira dimension; more precisely, one expects that, for every holomorphic map $f : \mathbf{C} \to X(\mathbf{C})$ to such a variety, the image $f(\mathbf{C})$ be contained in a proper closed algebraic subvariety.

Finally an (algebraic) function field analogue of Vojta's Conjecture predicts that, given a smooth curve $\tilde{\mathcal{C}}$ and a finite subset $S \subset \tilde{\mathcal{C}}$, there should exist a bound for the degree (in a suitable projective embedding) for the images of non-constant morphisms $\tilde{\mathcal{C}} \setminus S \to X$, where X is again an algebraic variety with maximal log Kodaira dimension.

The particular case where $\tilde{X}$ is the projective plane has been widely studied, in the arithmetic, analytic and algebraic setting. The condition on the Kodaira dimension of $D + K$ is equivalent to the inequality $\deg(D) \geq 4$. In all settings, it turns out that the case where D has at least four components is much easier: in this case, and only in this case, X embeds into the torus $\mathbf{G}_m^3$, in such a way that the image is a closed subvariety that is not the translate of a subtorus. Then the degeneracy of integral points is equivalent to the three-variable S-unit Theorem, which follows easily from the Subspace Theorem in diophantine approximation; the degeneracy of entire curves is a consequence of a famous theorem of Emile Borel (see [Bo]); finally, the function field analogue is obtained as an application of the Stothers-Mason *abc* Theorem (see [BrM]).

Hence the first crucial open case occurs with $\deg(D) = 4$ and D consisting of three irreducible components, i.e. D is the sum of a conic and two lines. In the arithmetic case, first considered by Beukers [Be, p. 116], the solution seems to be, at present, beyond hope (but some particular cases have been settled in [CZ1]). An open question, which would have a positive answer under Vojta's conjecture for the complement

of a conic and two lines, is the finiteness of the perfect squares of the form $2^a + 3^b + 1$ for positive integers a, b.

The Nevanlinna-theoretic analogue has been investigated since the seventies: Mark Green attacked a particular case, which amounts to the characterisation of squares, in the ring of entire functions, of the form $\exp(\varphi) + \exp(\psi) + 1$ for entire functions φ, ψ; he characterises such functions under technical hypothesis on the growth of φ, ψ. Recently, Noguchi, Winkelmann and Yamanoi solved completely the problem, even in much greater generality, by classifying in particular the entire curves on the complement of a conic and two lines.

The present paper solves the (algebraic) function field analogue, which to us seems to be nontrivial already in the case of rational curves.

We now describe precisely our main result. We let κ be an algebraically closed field of characteristic zero; all algebraic varieties will be defined over κ. We let $X = \mathbf{P}_2 \setminus D$, where D is a quartic consisting of the union of a smooth conic and two lines in general position (i.e. D has five singular points). Let $\mathcal{C} \subset X$ be an affine curve on X; its normalization is of the form $\tilde{\mathcal{C}} \setminus S$, for a (unique) smooth complete curve $\tilde{\mathcal{C}}$ and a finite subset $S \subset \tilde{\mathcal{C}}$. We define the *Euler characteristic* of $\mathcal{C}$ to be the Euler characteristic of $\tilde{\mathcal{C}} \setminus S$, i.e.

$$\chi(\mathcal{C}) = \chi_S(\tilde{\mathcal{C}}) = 2g(\tilde{\mathcal{C}}) - 2 + \sharp S.$$

This definition is consistent with Def. 1.2 of [KMK], where it is said that a curve $\bar{\mathcal{C}}$ meets k times a divisor D if the normalization of $\bar{\mathcal{C}} \setminus (D \cap \bar{\mathcal{C}})$ has k points at infinity.

Finally, we let $\deg(\mathcal{C})$ be the projective degree of the projective closure $\bar{\mathcal{C}}$ of $\mathcal{C}$ in $\mathbf{P}_2$. Our first result is

Theorem 1.1. *Let $X = \mathbf{P}_2 \setminus D$ be as before, $\mathcal{C} \subset X$ an affine curve. Then*

$$\deg(\mathcal{C}) \leq 2^{15} \cdot 35 \cdot \max\{1, \chi(\mathcal{C})\}. \tag{1.1}$$

Theorem 1.1 will be proved after reducing it (via Lemma 3.3) to the "diophantine equation"

$$y^2 = u_1^2 + \lambda u_1 + u_2 + 1, \tag{1.2}$$

to be solved in rational functions $(y, u_1, u_2) \in \kappa(\tilde{\mathcal{C}})$ where y is regular on $\tilde{\mathcal{C}} \setminus S$ and u_1, u_2 are S-units, i.e. regular and never vanishing functions on $\tilde{\mathcal{C}} \setminus S$ (see Theorem 3.4). A natural generalization, which will be obtained here by the same method, concerns more general bounds for the number of multiple zeros of rational functions of the form $A(u_1, u_2)$ for polynomials $A(X, Y) \in \kappa[X, Y]$. We shall prove the following generalization of Theorem 3.4, where the symbol $H(\cdot)$ is used to denote the *height* of a rational function on $\tilde{\mathcal{C}}$, i.e. its degree as a morphism $\tilde{\mathcal{C}} \to \mathbf{P}_1$.

Theorem 1.2. *Let $\tilde{\mathcal{C}}$ be a smooth complete curve, $S \subset \tilde{\mathcal{C}}$ a finite set of points. Let $A(X, Y) \in \kappa[X, Y]$ be a polynomial without repeated factors. For every positive ϵ there exists an integer $H = H(A, \epsilon, \chi(\tilde{\mathcal{C}} \setminus S))$ such that for all pairs $(u_1, u_2) \in (\mathcal{O}_S^*)^2$ with $\max\{H(u_1), H(u_2)\} \geq H$, either the regular function $A(u_1, u_2)$ has at most $\epsilon \max\{H(u_1), H(u_2)\}$ multiple zeros outside S, or u_1, u_2 verify a multiplicative dependence relation of the form*

$$u_1^r \cdot u_2^s \in \kappa \tag{1.3}$$

for a suitable pair of integers $(r, s) \in \mathbf{Z}^2 \setminus \{0\}$.

More precisely, we may choose $H(A, \epsilon, \chi(\tilde{\mathcal{C}} \setminus S)) \ll_{A,\epsilon} \chi(\tilde{\mathcal{C}} \setminus S)$.

Plainly, Theorem 1.2 applies in particular to the equation

$$y^d = A(u_1, u_2), \qquad d \geq 2, \tag{1.4}$$

to be solved in S-units $u_1, u_2 \in \mathcal{O}_S^*$ and S-integers $y \in \mathcal{O}_S$. As a result, we obtain a bound for the height of solutions to (1.4), outside some possible infinite family satisfying (1.3).

An immediate corollary of Theorem 1.2 (in the above quantification), which we do not state for simplicity, is the degeneracy of the solutions for a Fermat-type equation

$$z^d = A(x^n, y^m),$$

for any $d \geq 2$ and n, m large enough in terms of $\deg A$. To get a proof, it suffices to note that x^n, y^m are S-units for a set S which is small with respect to the involved degrees.

Any bound for the height of the solutions to such equations (1.4), in turn, corresponds to a bound for the degree of an affine curve, in terms of its Euler characteristic, on affine varieties obtained as ramified cyclic covers of $\mathbf{G}_m^2$. Note that the complement of a conic and two lines in $\mathbf{P}_2$ admits a finite map of degree two to $\mathbf{G}_m^2$, i.e. X is a double ramified cover of $\mathbf{G}_m^2$. Theorem 1.2 admits the following consequence:

Theorem 1.3. *Let Y be a smooth affine surface with log Kodaira dimension two. Suppose that there exists a cyclic (ramified) cover $Y \to \mathbf{G}_m^2$. Let us embed Y in a projective space $\mathbf{P}_N$. There exists a positive number c, depending on Y and its projective embedding such that for every curve $\mathcal{C} \subset Y$,*

$$\deg(\mathcal{C}) \leq c \cdot \max\{1, \chi(\mathcal{C})\}.$$

We observe that the condition that Y is a ramified cover of $\mathbf{G}_m^2$ appears also in the main theorem of [NWY], where however the cyclicity condition is not assumed and the torus $\mathbf{G}_m^2$ might be replaced by any arbitrary semi-abelian variety. The authors prove in [NWY] that under the same hypothesis on the log Kodaira dimension every entire curve on Y is degenerate.

We shall prove in §4 that no result like Theorem 1.1 can hold if the divisor D is a reducible cubic instead of a reducible quartic (in which case the log canonical divisor of the complement would be zero).

Theorem 1.4 (Counter-example). *Let $D \subset \mathbf{P}_2$ be a plane reducible cubic with normal crossing singularities. There exist curves $\mathcal{C} \subset \mathbf{P}_2 \setminus D$ of arbitrary high degree but with vanishing Euler characteristic.*

When the divisor D is an *irreducible* cubic, Beukers proved [Be, Thm. 3.3] in the arithmetic case, that the integral points on $\mathbf{P}_2 \setminus D$ are Zariski-dense, at least in a suitable extension of the ring of integers. Beukers method also holds over function fields, but in that case does not provide automatically integral points of arbitrary high height.

A possible generalization of Theorem 1.2 (so also of Theorems 1.1. and 1.3) concerns the case of polynomials $A(X, Y) \in \kappa(\tilde{\mathcal{C}})[X, Y]$, i.e. with non-constant coefficients. Geometrically, this situation corresponds to the data of a three-fold X and a projection $\theta : X \to \mathcal{C}$. The solutions (u_1, u_2, y) to (1.3) correspond to sections of θ. The method developed in the present paper permits to obtain the analogue results in this setting too.

Another application of the present method concerns rational curves on diagonal or generalized Fermat hypersurfaces. For simplicity here we limit ourselves to the above statements, leaving such generalizations to a future paper.

This paper is organized as follows: in the next section, we shall prove a diophantine estimate (Theorem 2.1, Theorem 2.2 and its Corollary) concerning the "greatest common divisor" of two rational functions on $\tilde{\mathcal{C}}$ of the form $a - 1, b - 1$, where a, b have all their zeros and poles in a prescribed set. This result, which is the function field analogue of Proposition 2 of [CZ2], is of independent interest and admits further applications which will be developed elsewhere.

In §3 we shall apply such estimates to bound the degree of the solutions of equations of the form $y^2 = u_1^2 + \lambda u_1 + u_2 + 1$, where u_1, u_2 are, as above, rational functions with prescribed zero and pole sets, and λ is an arbitrary scalar. As a consequence, we shall obtain the inequality of Theorem 3.4 concerning the solutions to equation (1.2). Theorem 1.1, which is reformulated in equivalent form as Theorem 3.1, is deduced from Theorem 3.4. The proof of Theorems 1.2 (which is a generalization of Theorem 3.4) and 1.3 (generalizing Theorem 1.1) runs along the same lines; a quick argument will be given at the end of §3. In §4 we prove that the hypothesis of Theorem 1.1 cannot be weakened, neither by taking for the divisor

D a reducible curve of degree ≤ 3 (instead of 4), nor by removing the hypothesis on the normal crossing singularities. Finally, in an appendix we shall show some arithmetic result in the spirit of Liardet-Raynaud theorems on torsion points on curves.

Sketch of the proofs. For the reader's convenience, we now give here a brief outline of the proof of Theorem 1.1. First of all an affine curve $\mathcal{C}$ in the surface $X = \mathbf{P}_2 \setminus D$ corresponds to a morphism from the normalization $\tilde{\mathcal{C}} \setminus S$ of $\mathcal{C}$ to X. Such a morphism, in turn, corresponds to the solution of the "diophantine equation" (1.2) in rational functions $y, u_1, u_2 \in \kappa(\tilde{\mathcal{C}})$ with u_1, u_2 S-units (see Lemma 3.3). We write (1.2) as $y^2 = A(X, Y)$ after setting $A(X, Y) = X^2 + \lambda X + Y + 1$; differentiating both sides (with respect to a suitable differential operator to be introduced in Lemma 3.5) one gets another equation satisfied by y, of the form $2yy' = B(u_1, u_2)$ where the polynomial $B(X, Y) = \frac{u_1'}{u_1} X \frac{\partial A}{\partial X} + \frac{u_2'}{u_2} Y \frac{\partial A}{\partial Y}$ has its coefficients in the field $\kappa(\tilde{\mathcal{C}})$. It turns out that the height of such coefficients is bounded in terms of S only, so in particular it is independent of the height of u_1, u_2, y. Now we obtain that the values of the two polynomials $A(X, Y), B(X, Y)$, calculated at the S-unit points u_1, u_2, have "many" zeros in common, i.e. the zeros of y. These facts can be reformulated in terms of a functional greatest common divisor.

Let us consider for instance the particular case $A(X, Y) = 1 + X + Y$ (so the equation (1.2) becomes $y^2 = 1 + u_1 + u_2$). From $2yy' = u_1' + u_2' = \frac{u_1'}{u_1} u_1 + \frac{u_2'}{u_2} u_2$ one gets

$$\frac{u_1'}{u_1} y^2 - 2yy' = \frac{u_1'}{u_1} + \left(\frac{u_1'}{u_1} - \frac{u_2'}{u_2}\right) u_2 = -\frac{u_1'}{u_1}\left(u_2\left(\frac{u_2'}{u_2} - \frac{u_1'}{u_1}\right)\frac{u_1}{u_1'} - 1\right) = -\frac{u_1'}{u_1}(w_2 - 1)$$

where $w_2 = u_2(\frac{u_1'}{u_1}\frac{u_2'}{u_2} - 1)$. Also by symmetry

$$\frac{u_2'}{u_2} y^2 - 2yy' = -\frac{u_2'}{u_2}(w_1 - 1)$$

with $w_1 = u_1(\frac{u_1'}{u_1}\frac{u_2'}{u_2} - 1)$. The rational functions w_1, w_2 are S'-units for a finite set S' which depends on S (and of course the chosen derivation $'$), but not on u_1, u_2. Then we are in the function-field analogue of the situation studied in [CZ2], where we proved a bound for $\gcd(w_1 - 1, w_2 - 1)$ for units w_1, w_2 in a number field. (We note that this bound is directly linked to Vojta's conjecture on diophantine approximation on certain surfaces, as remarked by Silverman [S].) As in the arithmetic case, we obtain such a bound, actually in stronger form; this is done in the next paragraph. Reinterpreting this inequality in terms of the morphism corresponding to (u_1, u_2, y), we obtain a bound for the degree of the image of such morphisms, consisting in the inequality (1.1) of Theorem 1.1.

§2. Greatest common divisors over function fields.

Let κ be, as before, an algebraically closed field of characteristic zero, $\tilde{\mathcal{C}}$ a smooth complete curve defined over κ, of genus $g = g(\tilde{\mathcal{C}})$, $S \subset \tilde{\mathcal{C}}$ a finite nonempty set of points of $\tilde{\mathcal{C}}$. We let $\mathcal{O}_S := \kappa[\tilde{\mathcal{C}} \setminus S]$ be the ring of S-integers, i.e. of regular functions on the affine curve $\tilde{\mathcal{C}} \setminus S$, and $\mathcal{O}_S^*$ be the group of S-units, i.e. of rational functions on $\tilde{\mathcal{C}}$ with all their zeros and poles in S. As usual, for a rational function $a \in \kappa(\tilde{\mathcal{C}})$, we let $H(a) = H_{\tilde{\mathcal{C}}}(a)$ be its height, i.e. its degree as a morphism $a : \tilde{\mathcal{C}} \to \mathbf{P}_1$. For $n \geq 2$ and elements $u_1, \ldots, u_n \in \kappa(\tilde{\mathcal{C}})$ not all zero, we denote by $H_{\tilde{\mathcal{C}}}(u_1 : \ldots : u_n)$, or simply $H(u_1 : \ldots : u_n)$, the projective height

$$H(u_1 : \ldots u_n) = H_{\tilde{\mathcal{C}}}(u_1 : \ldots : u_n) = -\sum_{v \in \tilde{\mathcal{C}}} \min\{v(u_1), \ldots, v(u_n)\}.$$

Also, for $u \in \kappa(\tilde{\mathcal{C}})$, we let

$$H_S(u) = -\sum_{v \notin S} \min\{0, v(u)\}$$

be the number of poles (with multiplicity) outside S. Plainly $0 \leq H_S(u) \leq H(u)$.

As in the previous section, the Euler characteristic

$$\chi = \chi(\tilde{\mathcal{C}} \setminus S) = \sharp(S) + 2g(\tilde{\mathcal{C}}) - 2$$

will often appear. We also note that if a nonconstant S-unit exists, as in our context, then necessarily $\sharp(S) \geq 2$, so $\chi \geq 0$.

We also make the following simple but important remark: if we replace the curve by a cover of it, of degree d, then the new heights will be multiplied by d while the new χ will be at least the old one multiplied by d. (This follows at once from the Riemann-Hurwitz formula, taking into account the possible ramifications in and out of S.) By this observation it will be clear that in proving the statements in this section we may often consider the minimal function field containing the relevant quantities.

Proposition 2.1. *Let $a, b \in \mathcal{O}_S^*$ be multiplicatively independent S-units, not both constant, and let h, k be positive integers. Then, either*

$$H(a) \leq h \cdot [\kappa(\tilde{\mathcal{C}}) : \kappa(a,b)], \qquad H(b) \leq k \cdot [\kappa(\tilde{\mathcal{C}}) : \kappa(a,b)]$$

or

$$H_S\left(\frac{1-a}{1-b}\right) \geq \frac{hk}{hk+h+k} H(b) - \frac{k}{hk+h+k}(H(a)+H(b)) - \frac{hk+h+k-1}{2}\chi.$$

Another, essentially equivalent, formulation of the result involves a kind of "$S-\gcd$" of $1-a, 1-b$, which here we measure by counting the number of common zeros outside S. Namely:

Proposition 2.2. *Let $a, b \in \mathcal{O}_S^*$ be multiplicatively independent S-units, not both constant, and let h, k be positive integers. Then, either*

$$H(a) \leq h \cdot [\kappa(\tilde{\mathcal{C}}) : \kappa(a,b)], \qquad H(b) \leq k \cdot [\kappa(\tilde{\mathcal{C}}) : \kappa(a,b)]$$

or

$$\sum_{v \notin S} \min\{v(1-a), v(1-b)\} \leq \frac{h+2k}{hk+h+k} H(b) + \frac{k}{hk+h+k} H(a) + \frac{hk+h+k-1}{2}\chi.$$

For the sake of generality, we have let here h, k be any positive integers, but special choices of them obviously may lead to simpler bounds; e.g., given $\epsilon > 0$, on choosing $h = k \sim 4\epsilon^{-1}$ we see that the right side of the displayed inequality in Proposition 2.2 becomes $\leq \epsilon(H(a)+H(b)) + O_{\sharp S, g, \epsilon}(1)$. More explicitly, we have for instance the following corollary, in which we also add a complementary part concerning the case of multiplicatively dependent a, b. In this case we have some nontrivial relation $a^r = \lambda b^s$ for integers r, s not both zero and a $\lambda \in \kappa^*$; if not both a, b are constant, such pairs (r, s) form a 1-dimensional lattice in $\mathbf{Z}^2$ and if (r, s) generates the lattice we speak of a generating relation. We have:

Corollary 2.3. *Let $a, b \in \mathcal{O}_S^*$ be S-units, not both constant, and let $H := \max\{H(a), H(b)\}$.*

(i) If a, b are multiplicatively independent, we have

$$\sum_{v \notin S} \min\{v(1-a), v(1-b)\} \leq 3\sqrt[3]{2}(H(a)H(b)\chi)^{\frac{1}{3}} \leq 3\sqrt[3]{2}(H^2\chi)^{\frac{1}{3}}.$$

(ii) If a, b are multiplicatively dependent, let $a^r = \lambda b^s$ be a generating relation. Then either $\lambda \neq 1$ and $\sum_{v \notin S} \min\{v(1-a), v(1-b)\} = 0$, or $\lambda = 1$ and

$$\sum_{v \notin S} \min\{v(1-a), v(1-b)\} \leq \min\left\{\frac{H(a)}{|s|}, \frac{H(b)}{|r|}\right\} \leq \frac{H}{\max\{|r|, |s|\}}.$$

The constant $3\sqrt[3]{2}$ in part (i) cannot be replaced by anything smaller than $(4/3)^{1/3}$, as shown by the example $a = t^3$, $b = -t(t+1)$.

It is amusing that this corollary, which concerns function fields and is proved here by function field-methods, in turn easily implies the (well-known) arithmetical conclusion of the finiteness of torsion points on an irreducible curve in $\mathbf{G}_m^n$ which is not a translate of an algebraic subgroup. In an Appendix we shall briefly show this deduction, and also some relations of the results with papers by Ailon-Rudnick [AR] (on $\gcd(f^n-1, g^n-1)$ for polynomials f, g) and Bombieri-Masser-Zannier [BMZ] (on the intersections of a given curve in $\mathbf{G}_m^n$ with the algebraic subgroups).

Proof of Proposition 2.1. We begin with some preliminary observations.

We have already noted that we may increase the function field to a finite extension without loss of generality. Therefore (since a, b are not both constant) we may assume that $\kappa(\tilde{\mathcal{C}}) = \kappa(a, b)$, namely we may assume that a, b generate the function field.

Since a, b are not both constant, $\sharp S \geq 2$, so $\chi \geq 0$; hence if $b \in \kappa$ the result holds trivially. Let us then suppose that $b \notin \kappa$.

Let now $c = 1 - b$, so $b + c = 1$, and let S' be the set of zeros of c outside S. Since b, c are nonconstant $(S \cup S')$-units, by Mason's *abc*-theorem for function fields (e.g., Cor. 1 of Thm. A of [BrM] with $n = 3$) we have $H(c) \leq H(1 : b : c) \leq \sharp S + \sharp S' + 2g - 2$. But $H(c)$ is the number of zeros of c, counted with multiplicity, so in particular $\sum_{v(c)>0} v(c) \leq \sharp S + \sharp S' + 2g - 2$. Since $v(c) > 0$ for all $v \in S'$, we have

$$\sum_{v(1-b)>0} (v(1-b) - 1) \leq \chi. \tag{2.1}$$

Next, suppose we have a nontrivial relation $a^r b^s = \lambda \in \kappa^*$. If $a - 1$ and $b - 1$ have no common zero, we have

$$H_S\left(\frac{1-a}{1-b}\right) \geq \sum_{v(1-b)>0, v \notin S} v(1-b) = H(1-b) - \sum_{v(1-b)>0, v \in S} v(1-b).$$

The first term on the right equals $H(b)$; also, $0 \leq \sum_{v(1-b)>0, v\in S} v(1-b) \leq \sharp S - 2 + \sum_{v(1-b)>1, v \in S}(v(1-b)-1)$. Hence by (2.1) we have $H_S(\frac{1-a}{1-b}) \geq H(b) - (2\sharp S + 2g - 4)$, proving the proposition and more. Hence we may assume that $1-a, 1-b$ have some common zero P. Then $a(P) = b(P) = 1$, which yields $\lambda = 1$, against the multiplicative independence assumption.

Therefore in what follows we shall assume that a, b are multiplicatively independent modulo κ^*.

The argument now will mimic [CZ2], but will involve a simple Wronskian argument rather than the Schmidt Subspace Theorem. (Especially after [BrMa], Wronskians are a familiar tool in diophantine questions over function fields. See also [W] for an effective proof with Wronskians of a function field version of the Subspace Theorem.) Thus we recall some standard facts about Wronskians.

For elements $f_1, \dots, f_n \in K$ and a nonconstant $t \in K$ we let the "Wronskian" $W_t(f_1, \dots, f_n)$ be the determinant of the $n \times n$ matrix whose j-th row-entries are the $(j-1)$-th derivatives of the f_i's with respect to t. It is well known that $W_t = 0$ if and only if the f_i's are linearly dependent over κ. (Recall that here char $\kappa = 0$.) Let $z \in K$ be another nonconstant element. Then we have the known, easily proved, formula

$$W_z(f_1, \dots, f_n) = (\frac{dt}{dz})^{\binom{n}{2}} W_t(f_1, \dots, f_n). \tag{2.2}$$

For a place v of K we choose once and for all a local parameter t_v at v and we define $W_v := W_{t_v}$. This depends on the choice of t_v, but (2.2) shows that the order $v(W_v)$ depends only on v.

To prove the Proposition we shall consider suitable Wronskians.

We define $q = (1-a)/(1-b)$ and, letting $n := hk + h + k$, we define functions $f_1, \dots, f_n$ as follows. For $i = 1, \dots, k$ we let $f_i := a^{i-1} q$ while we define $f_{k+1}, \dots, f_n$ as the functions $a^r b^s$, $r = 0, 1, \dots, k$, $s = 0, 1, \dots, h-1$, in some order.

We now choose nonconstant t, t_v as above and we put

$$\omega = W_t(f_1, \dots, f_n), \qquad \omega_v = W_{t_v}(f_1, \dots, f_n).$$

Suppose first that $\omega = 0$; then, as we have remarked, the f_i are linearly dependent over the constant field κ; recalling the definition of the f_i this amounts to a relation $P_1(a)(1-a) + P_2(a,b)(1-b) = 0$, where $P_1(X)$ is a polynomial of degree $\leq k-1$ and $P_2(X,Y)$ is a polynomial of degree $\leq k$ in X and $\leq h-1$ in Y and where not both P_1, P_2 are zero. Observe that $P_1(X)(1-X) + P_2(X,Y)(1-Y)$ is not identically zero, for otherwise $P_1(X)$ would vanish (set $Y = 1$) and then P_2 would also vanish, a contradiction. Then we find a nontrivial polynomial relation $P(a,b) = 0$, for a polynomial $P \neq 0$ of degree $\leq k$ in X and $\leq h$ in Y. Of course we may assume that P is irreducible.

Then, since a, b generate our function field, we have $H(a) = \deg a \leq h$, $H(b) = \deg b \leq k$, falling into the first possibility of the sought conclusion.

Therefore in what follows we assume that $\omega \neq 0$.

To go on, we first seek for a lower bound for $v(\omega_v)$ and for this we shall distinguish several cases.

Case (i): $v \notin S$, $v(q) < 0$. We first make a simple observation: suppose that two functions f_i, f_j, $i \neq j$, have a pole at v of the same order > 0. Then by subtracting from the i-th column a constant multiple of the j-th column, we may suppose in calculating the Wronskian that f_i has a pole of smaller order than f_j at v. Therefore, by repeating this procedure we may assume that the functions f_i which have a pole at v have in fact poles of pairwise distinct orders, not exceeding the original maximal order.

In the present Case (i) the only functions which (may) have a pole at v are $f_1, \ldots, f_k$, because $v \notin S$, so a and the remaining f_i's are units at v by assumption. Hence, the only poles we shall possibly find in ω_v will come from the first k columns. However by the above observation we may change the actual f_i's, $i = 1, \ldots, k$, to assume that the negative ones among $v(f_1), \ldots, v(f_k)$ are all distinct and $\geq v(q)$. Suppose that after such column operations and suitable renumbering only $f_1, \ldots, f_r$ have a pole at v and $v(f_1) < \ldots < v(f_r) < 0$. Plainly we will have $v(q) \leq v(f_1)$ and $r \leq k$, so $r \leq \min(k, -v(q))$. Also, observe that each derivation with respect to t_v increases the order of a pole by 1 and leaves regular a regular function at v. In conclusion, by looking at the individual terms obtained in the expansion of the determinant (after having performed the column operations), a simple inspection shows that we have $v(\omega_v) \geq v(f_1) + \ldots + v(f_r) + \binom{r}{2} - r(n-1)$, whence, since we are assuming the $v(f_i)$ to be distinct,

$$v(\omega_v) \geq rv(q) - r(n-r) \geq rv(q) + v(q)(n-r) = v(q)n.$$

Case (ii): $v \notin S$, $v(q) \geq 0$. Now every element of the local Wronskian matrix is v-integral, so the same holds for the determinant, i.e., $v(\omega_v) \geq 0$ in this case.

Case (iii): $v \in S$, $v(b) > 0$. This case contains the crucial point. As in [CZ2], we consider the identity

$$a^j q - a^j(1-a)(1 + b + \ldots + b^{h-1}) = a^j b^h q. \tag{2.3}$$

This will be useful to approximate $a^j q$ with a polynomial in a, b, at the places under consideration.

In fact, we may use the identity to replace, for $i = 1, \ldots, k$, the function f_i with the left side of (2.3), with $j = i-1$, which by (2.3) equals $a^{i-1}b^h q$, denoted g_i. Observe that this corresponds to subtract from f_i a certain κ-linear combination of $f_{k+1}, \ldots, f_n$, and thus the value of ω_v is unchanged. We have $v(g_i) = (i-1)v(a) + hv(b) + v(q)$. Since $v(\frac{d^l f}{dt_v^l}) \geq v(f) - l$, we easily find, on looking again at the individual terms in the determinant expansion, that

$$v(\omega_v) \geq \frac{k(k-1)}{2} v(a) + hkv(b) + kv(q) + \left(\sum_{i=k+1}^{n} v(f_i)\right) - \binom{n}{2}.$$

Case (iv): $v \in S$, $v(b) \leq 0$. We now argue directly with the terms in the determinant expansion (that is, we do not perform any column operation). Since $v(f_i) = (i-1)v(a) + v(q)$ for $i = 1, \ldots, k$, we find as in the previous case that

$$v(\omega_v) \geq \frac{k(k-1)}{2} v(a) + kv(q) + \left(\sum_{i=k+1}^{n} v(f_i)\right) - \binom{n}{2}.$$

Summing over all places v of $\kappa(\tilde{\mathcal{C}})$, taking into account the estimates obtained in the four cases, and recalling that $\sum_{v\in S} v(a) = \sum_{v\in S} v(b) = \sum_{v\in S} v(f_i) = 0$ for $i > k$, because a, b are S-units, we thus get

$$\sum_v v(\omega_v) \geq \sum_{v\notin S, v(q)<0} nv(q) + hk \sum_{v\in S, v(b)>0} v(b) + k\sum_{v\in S} v(q) - \binom{n}{2}\sharp S. \tag{2.4}$$

Now, $\sum_{v\in S, v(b)>0} v(b) = \sum_{v(b)>0} v(b)$, because b is an S-unit; also, $\sum_{v(b)>0} v(b) = -\sum_{v(b)<0} v(b) = H(b)$. Moreover, (2.2) with $z = t_v$ shows that $v(\omega_v) = \binom{n}{2} v(dt/dt_v) + v(\omega)$. On summing over v this yields

$$\sum_v v(\omega_v) = \binom{n}{2}\sum_v v(\frac{dt}{dt_v}) + \sum_v v(\omega) = \binom{n}{2}(2g-2),$$

the last equality holding because of the product formula (for $\omega \in \kappa(\tilde{\mathcal{C}})^*$) and because $2g-2$ is the degree of any canonical divisor. Comparing with the above yields

$$\binom{n}{2}\chi - \sum_{v\notin S, v(q)<0} nv(q) \geq hkH(b) + k\sum_{v\in S} v(q). \tag{2.5}$$

Finally, $-\sum_{v\in S} v(q) \leq H(q) \leq H(a) + H(b)$, whence

$$H_S(q) \geq \frac{hk}{hk+h+k}H(b) - \frac{k}{hk+h+k}(H(a)+H(b)) - \frac{n-1}{2}\chi,$$

concluding the proof.

Proof of Proposition 2.2. Note that by definition $H_S(\frac{1-a}{1-b}) = \sum_{v\notin S, v(1-b)>v(1-a)}(v(1-b) - v(1-a))$. In turn the right side may be plainly replaced by $\sum_{v\notin S}(v(1-b) - \min(v(1-b), v(1-a)))$.

Now, $\sum_{v\notin S} v(1-b)$ does not exceed $H(1-b) = H(b)$, whence

$$H_S(\frac{1-a}{1-b}) \leq H(b) - \sum_{v\notin S}\min\{v(1-a), v(1-b)\}.$$

An application of Proposition 2.1 now immediately leads to the sought inequality.

Proof of Corollary 2.3. If one of a, b is constant, then it cannot be 1 because a, b are multiplicatively independent. Therefore the sum vanishes and we are done. So, let us suppose that none of a, b is constant. Then we may also assume that $H(a) \geq H(b) > 0$.

Let us first deal with part (i), supposing that $\chi = 0$ to start with; then $\#S = 2, g = 0$ and necessarily there is a relation $a^r b^s = \gamma \in \kappa^*$, where r, s are not both zero: this is because some function $a^{\deg b} b^{\pm \deg a}$ has no zeros or poles. Then $\gamma \neq 1$ by assumption, so $\min(v(1-a), v(1-b)) \leq 0$ for all v, proving the result.

Therefore, suppose $\chi \neq 0$ in the sequel.

Also, as we have remarked, we may assume that $\kappa(\mathbf{C}) = \kappa(a, b)$.

Let us choose now $h = [(4H(a)^2/H(b)\chi)^{1/3}] - 1$, $k = [(4H(b)^2/H(a)\chi)^{1/3}] - 1$.

Suppose that $k < 1$. Then $[(4H(b)^2/H(a)\chi)^{1/3}] < 2$, whence $(4H(b)^2/H(a)\chi)^{1/3} < 2$ and so $H^2(b) < 2H(a)\chi$. Then $H^3(b) < 2H(a)H(b)\chi$.

In this case we use the obvious inequality $\sum_{v\notin S}\min(v(1-a), v(1-b)) \leq H(1-b) = H(b) < (2H(a)H(b)\chi)^{1/3}$, proving what we need, with a better constant.

Hence in what follows we assume that $h \geq k \geq 1$, so in particular we may apply Proposition 2.2. The conclusion gives two possibilities.

In the first case we have $H(a) \leq h$, whence $H(a)^3 \leq (h+1)^3 \leq 4H(a)^2/H(b)\chi$, whence $H(a)H(b)\chi \leq 4$. Since $\chi \geq 1$, this implies $H(b) \leq 2$ so as above we obtain $\sum_{v\notin S}\min(v(1-a), v(1-b)) \leq H(1-b) = H(b) \leq 2$. Again this gives the sought result since $H(a)H(b)\chi \geq 1$.

Then we may assume $H(a) > h$, so the second alternative of Proposition 2.2 must hold.

It is easily checked that the coefficient $(h+2k)/(hk+h+k)$ is bounded by $3/(k+2)$, since $h \geq k$. Similarly, $k/(hk+h+k) \leq 1/(h+2)$. Therefore

$$\sum_{v \notin S} \min(v(1-a), v(1-b)) \leq \frac{3}{k+2} H(b) + \frac{1}{h+2} H(a) + \frac{(h+1)(k+1)-2}{2} \chi.$$

We now use $k+2 \geq (4H(b)^2/H(a)\chi)^{1/3}$, $h+2 \geq (4H(a)^2/H(b)\chi)^{1/3}$ and also $(h+1)(k+1) \leq (4H(b)^2/H(a)\chi)^{1/3}(4H(a)^2/H(b)\chi)^{1/3} = (16H(a)H(b)/\chi^2)^{1/3}$, which yields

$$\sum_{v \notin S} \min(v(1-a), v(1-b)) \leq 4(\frac{H(a)H(b)\chi}{4})^{1/3} + \frac{1}{2}(16H(a)H(b)\chi)^{1/3} = \chi < 3\sqrt[3]{2}(H(a)H(b)\chi)^{1/3}$$

concluding the proof of part (i).

For part (ii), if $\lambda \neq 1$ the result follows as in part (i), so suppose $a^r = b^s$ in a generating relation. Plainly we have that $\gcd(r,s) = 1$ whence $hr + ks = 1$ for some integers h, k. Then, setting $c := a^k b^h$, which is an S-unit, we have $a = c^s, b = c^r$. This yields $\min(v(1-a), v(1-b)) = v(1-c) + \min(v(\frac{1-c^s}{1-c}), v(\frac{1-c^r}{1-c}))$. This last term vanishes for $v \notin S$, since then c is regular at v and the polynomials $\frac{1-X^s}{1-X}$ and $\frac{1-X^r}{1-X}$ are coprime. Therefore the relevant sum equals $\sum_{v \notin S} v(1-c) \leq H(c)$ and the result follows at once since $|s|H(c) = H(a)$, $|r|H(c) = H(b)$.

§3. Maps to the complement of a conic and two lines.

Let us consider the configuration of a conic and two lines in $\mathbf{P}_2$: let D_1 be a smooth conic in $\mathbf{P}_2$, D_2, D_3 be distinct lines, all defined over κ. We are particularly interested in the case where D_1, D_2, D_3 are in general position, by this meaning that the two lines D_2, D_3 intersect the conic D_1 at four distinct points. Such configurations form a one-dimensional family, parametrized by the cross-ratio of the four intersection points in $D_1 \cap (D_2 \cup D_3)$ with respect to the conic.

Let now $\tilde{\mathcal{C}}$ be a smooth complete algebraic curve of genus $g(\tilde{\mathcal{C}})$ defined over κ, and $S \subset \tilde{\mathcal{C}}$ be a finite nonempty set of points of $\tilde{\mathcal{C}}$. We shall denote again by $\mathcal{O}_S$ the ring $\kappa[\tilde{\mathcal{C}} \setminus S]$ of regular functions on the affine curve $\tilde{\mathcal{C}}$; its elements will also be called S-integers; then $\mathcal{O}_S^*$ will be its group of units, i.e. the multiplicative group of rational functions on $\tilde{\mathcal{C}}$ having all their zeros and poles in S; they will be called S-units. We are interested in classifying regular maps from $\tilde{\mathcal{C}} \setminus S$ to $\mathbf{P}_2$ "omitting" the divisor $D := D_1 + D_2 + D_3$, i.e. morphisms $f : \tilde{\mathcal{C}} \to \mathbf{P}_2$ such that $f^{-1}(D) \subset S$.

Before passing to the "general position" situation, i.e. when the lines D_2, D_3 intersect the conic D_1 at four distinct points we analyse some "degenerate" cases which deserve attention.

A first interesting case comes from a tangent line D_2 to the conic D_1 and a line D_3 with $D_1 \cap D_2 \cap D_3 = \emptyset$. This case is easily reduced to the "diophantine equation"

$$1 + u_1 + u_2 = y^2 \tag{3.1}$$

to be solved in units $u_1, u_2 \in \mathcal{O}_S^*$ and regular functions $y \in \mathcal{O}_S$.

This situation can also be recovered from the "general position" case where the four intersection points of $(D_2 \cup D_3) \cap D_1$ have cross-ratio -1 with respect to the conic D_1 (the reduction can be obtained via the unramified covering $v^2 = u_2$). The special case where moreover $\mathcal{C} = \mathbf{P}_1$ has genus zero and S has three points has been treated by the second author in [Z2], using different tools.

A second degenerate case is provided by two distinct non tangent lines D_2, D_3 to the conic D_1 such that $D_1 \cap D_2 \cap D_3$ is non empty (so necessarely consists of a single point). The problem in this case can be reduced to the equation

$$y = \frac{u_1 - 1}{u_2 - 1}$$

to be solved in S-units u_1, u_2 and S-integers $y \in \mathcal{O}_S$. So it consists of a pure divisibility problem in the ring $\mathcal{O}_S$. In this case, however, no bound of the form (1.1) holds on the corresponding surface $\mathbf{P}_2 \setminus D$ (this is the content of Proposition 4.3).

¿From now on we shall suppose that D_1, D_2, D_3 are in general position, and shall denote by D also the support of the divisor $D_1 + D_2 + D_3$. We begin by restating Theorem 1.1 in the following way.

Theorem 3.1. *Let $\tilde{\mathcal{C}}, S, D$ be as above. Let $f : \tilde{\mathcal{C}} \to \mathbf{P}_2$ be a non constant morphism such that $f^{-1}(D) \subset S$. Then the degree of the curve $f(\tilde{\mathcal{C}})$ verifies*

$$\deg(f(\tilde{\mathcal{C}})) \leq 2^{15} \cdot 35 \cdot \max\{1, \chi(\tilde{\mathcal{C}} \setminus S)\}. \tag{3.2}$$

Let us introduce the **notation for the proofs**. As in the previous paragraph, we associate with each point v of $\tilde{\mathcal{C}}$ a discrete valuation of the function field $\kappa(\tilde{\mathcal{C}})$, trivial on κ, normalized so that its value group is the group of integers $\mathbf{Z}$ and shall denot it by the same letter v. The height of a rational function $a \in \kappa(\tilde{\mathcal{C}})$ will be denoted now by the symbol $H_{\tilde{\mathcal{C}}}(a)$, since the reference to the curve $\tilde{\mathcal{C}}$ will be relevant in the sequel; recall that it coincides with the degree of a viewed as a morphism $\tilde{\mathcal{C}} \to \mathbf{P}_1$; hence it is given by the formula

$$H_{\tilde{\mathcal{C}}}(a) = \sum_{v \in \tilde{\mathcal{C}}} \max\{0, v(a)\}.$$

If $\tilde{\mathcal{D}} \to \tilde{\mathcal{C}}$ is a dominant morphism of smooth (irreducible) projective curves, then the function field of $\tilde{\mathcal{C}}$ injects into the function field of $\tilde{\mathcal{D}}$; in that case an element $a \in \kappa(\tilde{\mathcal{C}})$ could also be viewed as a rational function on $\tilde{\mathcal{D}}$. We shall then write $H_{\tilde{\mathcal{D}}}(a)$ to denote its degree as a function on $\tilde{\mathcal{D}}$; it verifies $H_{\tilde{\mathcal{D}}}(a) = [\kappa(\tilde{\mathcal{D}}) : \kappa(\tilde{\mathcal{C}})] \cdot H_{\tilde{\mathcal{C}}}(a)$. For every element $a \in \kappa(\tilde{\mathcal{C}})$, we write $d(a)$ for its differential. Finally, recall that the Euler characteristic of the affine curve $\tilde{\mathcal{C}} \setminus S$ is by definition the integer $\chi_S(\tilde{\mathcal{C}}) = \chi(\tilde{\mathcal{C}} \setminus S) = 2g(\tilde{\mathcal{C}}) - 2 + \sharp(S)$.

We begin by proving the following

Lemma 3.2. *Let $\tilde{\mathcal{C}}, S$ be as before, and $f : \tilde{\mathcal{C}} \setminus S \to \mathbf{A}^2$ be a regular map from the affine curve $\tilde{\mathcal{C}} \setminus S$ to the affine plane, given as $f(p) = (x(p), y(p))$ for S-integers $x, y \in \mathcal{O}_S$. Then the degree of the image $f(\tilde{\mathcal{C}})$ (viewed as embedded canonically in $\mathbf{P}_2$) verifies*

$$\deg(f(\tilde{\mathcal{C}})) \leq H_{\tilde{\mathcal{C}}}(x) + H_{\tilde{\mathcal{C}}}(y).$$

Proof. There exists a choice of scalars $(a, b, c) \in \kappa^3$ with $(a, b) \neq (0, 0)$ such that the line of equation $ax + by = c$ does not intersect the curve $f(\mathcal{C})$ (where as usual $\mathcal{C} := \tilde{\mathcal{C}} \setminus S$) at infinity nor at any singular point and is not tangent at any point (actually this is true for a generic choice of a, b, c). Then the degree of $f(\tilde{\mathcal{C}})$ is the number of intersection points of the affine curve $f(\mathcal{C})$ with such a line. This number is bounded by the number of solutions in $\tilde{\mathcal{C}} \setminus S$ to the equation $ax(p) + by(p) = c$; now the (regular) function $ax(p) + by(p)$ has height $\leq H_{\tilde{\mathcal{C}}}(x) + H_{\tilde{\mathcal{C}}}(y)$, so the estimate follows. □

The next lemma will rely the existence of morphisms $\tilde{\mathcal{C}} \setminus S \to X$ to the solutions of a certain diophantine equation over the function field $\kappa(\tilde{\mathcal{C}})$.

Lemma 3.3. *Suppose $f : \tilde{\mathcal{C}} \setminus S \to \mathbf{P}_2 \setminus D$ is a morphism. There exist a scalar $\lambda \in \kappa$, S-units $u_1, u_2 \in \mathcal{O}_S^*$ and an S-integer $y \in \mathcal{O}_S$ satisfying*

$$H_{\tilde{\mathcal{C}}}(y) + H_{\tilde{\mathcal{C}}}(u_1) \geq \deg(f(\tilde{\mathcal{C}}))$$

such that

$$y^2 = u_1^2 + \lambda u_1 + u_2 + 1. \tag{3.3}$$

Proof. We can certainly choose homogeneous coordinates $(x_0 : x_1 : x_2)$ in $\mathbf{P}_2$ such that the intersection $D_1 \cap D_2$ consists of the two points $(0:1:1), (0:1:-1)$, while the intersection $D_1 \cap D_3$ is formed by the points $(1:0:1), (1:0:-1)$. Then the lines D_2, D_3 are defined by the equations

$$D_2: \ x_0 = 0 \qquad \text{and} \qquad D_3: \ x_1 = 0$$

respectively. Also the conic D_1 must belong to the pencil of conics defined by the equation

$$x_2^2 - x_1^2 - \lambda x_0 x_1 - x_0^2 = 0, \tag{3.4}$$

for a suitable $\lambda \in \kappa$. We shall now put $x := x_1/x_0$, $y = x_2/x_0$, which are affine coordinates for the plane $\mathbf{P}_2 \setminus D_2 \simeq \mathbf{A}^2$. Note that the line D_3 has the equation $x = 0$ in such affine coordinates. Let now $f : \tilde{\mathcal{C}} \setminus S \to \mathbf{P}_2 \setminus D$ be a regular map. The fact that $f(\tilde{\mathcal{C}} \setminus S)$ avoids the line at infinity D_2 and the line D_3 means precisely that f can be written in affine coordinates as

$$f(p) = (u_1(p), y(p)) \qquad (p \in \mathcal{C} \setminus S)$$

for a regular function $y \in \mathcal{O}_S$ and a unit $u_1 \in \mathcal{O}_S^*$. Imposing the condition that the image of f avoids also the conic defined by (3.4) means that the regular function

$$u_2 := y^2 - u_1^2 - \lambda u_1 - 1$$

is in fact a unit. Finally, the bound $H_{\tilde{\mathcal{C}}}(y) + H_{\tilde{\mathcal{C}}}(u_1) \geq \deg(f(\tilde{\mathcal{C}}))$ follows from the preceding Lemma, concluding the proof. $\square$

We shall now work with equation (3.3). Our aim is to prove the following

Theorem 3.4. *Let $\tilde{\mathcal{C}}, S$ and λ be as before; then every solution $(y, u_1, u_2) \in \mathcal{O}_S \times (\mathcal{O}_S^*)^2$ of equation (3.3) satisfies one of the following conditions:*

(i) a sub-sum on the right term of (3.3) vanishes;

(ii) u_1, u_2 verify a multiplicative dependence relation of the form $u_1^r \cdot u_2^s = \mu$, where $\mu \in \kappa^*$ is a scalar and r, s are integers, not both zeros, and both ≤ 5;

(iii) $\max\{H_{\tilde{\mathcal{C}}}(u_1), H_{\tilde{\mathcal{C}}}(u_2)\} \leq 2^{20} \cdot 35 \cdot \chi_S(\tilde{\mathcal{C}})$.

The idea of the proof is that if the right-side term in (3.3) is a square, then it has "many" zeros in common with its "derivative", which can also be written as a linear combination (with non-constant coefficients) of S-units. To exploit this remark we should first specify the precise meaning of "derivative"; for this reason we state the following

Lemma 3.5. *There exists a differential form ω on $\tilde{\mathcal{C}}$ and a finite set $T \subset \tilde{\mathcal{C}}$ of cardinality $\sharp(T) = \max\{0, 2g(\tilde{\mathcal{C}}) - 2\}$ such that for every S-unit $u \in \mathcal{O}_S^*$ there exists an $(S \cup T)$-integer $\theta_u \in \mathcal{O}_{S \cup T}$ having only simple poles such that*

$$\frac{d(u)}{u} = \theta_u \cdot \omega, \qquad H_{\tilde{\mathcal{C}}}(\theta_u) \leq \chi_S(\tilde{\mathcal{C}}). \tag{3.5}$$

Proof. Suppose first the genus $g(\tilde{\mathcal{C}}) \neq 0$. Then there exists a regular differential form ω, having $2g(\tilde{\mathcal{C}}) - 2$ distinct zeros of multiplicity one. Let T be its zero set, which has naturally cardinality $2g(\tilde{\mathcal{C}}) - 2$. Let now $u \in \mathcal{O}_S^*$ be an S-unit. Then the only rational function θ_u on $\tilde{\mathcal{C}}$ satisfying (3.5) can have poles only at the zeros of ω or the poles of $d(u)/u$, hence it is an $(S \cup T)$ integer. The poles of $d(u)/u$ are necessarily contained in S and of multiplicity ≤ 1. It follows that the number of poles with multiplicity of θ_u is $\leq \sharp(S) + 2g(\tilde{\mathcal{C}}) - 2$ as wanted. Consider now the case of genus 0. Let v, w be distinct points on S (note that $\sharp(S) \geq 2$, otherwise all S-units would be constant) and let ω be a differential form having two simple poles in v, w and no zero. Then the rational function θ_u appearing in (3.5) can have poles only in $S \setminus \{v, w\}$, of multiplicity at most one, so its height is again $\leq \sharp(S) - 2 = \chi_S(\tilde{\mathcal{C}})$ as wanted. $\square$

We shall now fix such a differential form ω as in Lemma 3.5, and also the finite set of places T (which will be empty if the genus of $\tilde{\mathcal{C}}$ is zero or one, and of cardinality $2g(\tilde{\mathcal{C}})-2$ for $g(\tilde{\mathcal{C}})\geq 2$). For a rational function $a\in\kappa(\tilde{\mathcal{C}})$, we shall denote by a' the only rational function such that

$$d(a)=a'\cdot\omega;$$

in such a notation, the rational function θ_u appearing in the previous lemma equals u'/u. The analogous estimate for S-integers (instead of S-units) is

Lemma 3.6. *Let $a\in\mathcal{O}_S$ be an S-integer. Then we have*

$$d(a)=a'\cdot\omega$$

where a' is a $(S\cup T)$-integer whose height satisfies

$$H_{\tilde{\mathcal{C}}}(a')\leq H_{\tilde{\mathcal{C}}}(a)+\chi_S(\tilde{\mathcal{C}}).$$

Moreover for every place of $v\in S\cup T$, we have $v(a')\geq v(a)-1$.

Proof. The proof is analogous to the previous one. As before, we first treat the case of positive genus. The poles of a' are either the zeros of ω (and we have $2g(\tilde{\mathcal{C}})-2$ of them counting multiplicities) or the poles of a. If v is such a pole, then $v\in S$ since a is supposed to be an S-integer, and $v(a')=v(a)-1$ (if v is not a zero of ω). Then the difference between the number of poles with multiplicities of a' and a is bounded by $\sharp(S)+2g(\tilde{\mathcal{C}})-2$ as wanted. The genus zero case is treated in the same way as before. $\square$

Our next goal will be the proof of a formal identity for the derivative of the values of a polynomial

Lemma 3.7. *Let $A(X,Y)\in\kappa[X,Y]$ be a polynomial, u_1,u_2 be S-units in $\kappa(\tilde{\mathcal{C}})$; let $B(X,Y)\in\mathcal{O}_{S\cup T}[X,Y]$ be the polynomial defined by*

$$B(X,Y)=\frac{u_1'}{u_1}\cdot X\frac{\partial A}{\partial X}(X,Y)+\frac{u_2'}{u_2}\cdot Y\frac{\partial B}{\partial Y}(X,Y).$$

Then, with the above notation for the 1-form ω, the differential of $A(u_1,u_2)$ is

$$d(A(u_1,u_2))=B(u_1,u_2)\cdot\omega.$$

Note that by using the symbol $'$ as before we can rewrite the above identity as $(A(u_1,u_2))'=B(u_1,u_2)$.

Proof. Write $A(X,Y)=\sum_{i,j}a_{i,j}X^iY^j$, where the sum runs over a finite subset of $\mathbf{N}^2$, and the coefficients $a_{i,j}$ are scalars in κ. The differential of each monomial calculated in (u_1,u_2) is

$$d(a_{i,j}u_1^iu_2^j)=\left(a_{i,j}\cdot i\cdot u_1'u_1^{i-1}u_2^j+a_{i,j}\cdot j\cdot u_1^iu_2'u_2^{j-1}\right)\cdot\omega=a_{i,j}\left(\frac{u_1'}{u_1}\cdot i\cdot u_1^iu_2^j+\frac{u_2'}{u_2}\cdot j\cdot u_2'u_1^iu_2^j\right)\cdot\omega$$

and the lemma follows. $\square$

We now return to equation (3.3).

Lemma 3.8. *Let $A(X,Y)\in\kappa[X,Y]$ and $B(X,Y)\in\mathcal{O}_{S\cup T}(\tilde{\mathcal{C}})[X,Y]$ be the polynomials*

$$\begin{aligned}A(X,Y)&=X^2+\lambda X+Y+1\\ B(X,Y)&=2\frac{u_1'}{u_1}\cdot X^2+\lambda\frac{u_1'}{u_1}\cdot X+\frac{u_2'}{u_2}\cdot Y;\end{aligned}\tag{3.6}$$

let $F(X) \in \mathcal{O}_{S\cup T}[X], G(Y) \in \mathcal{O}_{S\cup T}[Y]$ be the resultants of $A(X,Y), B(X,Y)$ with respect to Y and X, i.e. the polynomials

$$\begin{aligned} F(X) &:= \left(\frac{u_2'}{u_2} - \frac{u_1'}{u_1}\right) X^2 + \lambda \left(\frac{u_2'}{u_2} - 1\right) X + \frac{u_2'}{u_2} = \mathrm{Res}_Y(A(X,Y), B(X,Y)), \\ G(Y) &:= 2\left(2\frac{u_1'}{u_1} - \frac{u_2'}{u_2}\right) X^2 + \left((\lambda^2-2)\frac{u_1'}{u_1}\frac{u_2'}{u_2} + (4-\lambda^2)\frac{u_1'}{u_1} + 4\left(\frac{u_1'}{u_1}\right)^2\right) X + 2(4-\lambda^2)\left(\frac{u_1'}{u_1}\right)^2 \\ &\quad = \mathrm{Res}_X(A(X,Y), B(X,Y)). \end{aligned} \tag{3.7}$$

For every solution $(y, u_1, u_2) \in \mathcal{O}_S \times (\mathcal{O}_S^*)^2$ of (3.3) we have

$$\begin{aligned} y^2 &= A(u_1, u_2) \\ 2yy' &= B(u_1, u_2). \end{aligned}$$

Moreover the S-integer y divides both $F(u_1)$ and $G(u_2)$ in the ring $\mathcal{O}_{S\cup T}$.

Proof. Of course, equation (3.3) is exactly $y^2 = A(u_1,u_2)$ and $B(X,Y) = \frac{u_1'}{u_1} X \frac{\partial}{\partial X} A(X,Y) + \frac{u_2'}{u_2} Y \frac{\partial}{\partial Y} B(X,Y)$, so by the above lemma we have $2yy' = B(u_1,u_2)$ as wanted.

To prove that y divides $F(u_1)$ (resp. $G(u_2)$) we show that $F(X)$ (resp. $G(Y)$) are linear combinations of $A(X,Y), B(X,Y)$ with coefficients in $\mathcal{O}_{S\cup T}[Y]$ (resp. $\mathcal{O}_{S\cup T}[X]$). This is a general fact in the theory of resultants; in this case the linear combination $\frac{u_2'}{u_2} A(X,Y) - B(X,Y)$ equals $F(X)$, proving the first claim. The polynomial $G(Y)$ is obtained as the linear combination

$$\begin{aligned} & 2\frac{u_1'}{u_1}\left(2u_2\left(2\frac{u_1'}{u_1} - \frac{u_2'}{u_2}\right) + (4-\lambda^2)\frac{u_1'}{u_1} - \lambda u_1\right) \cdot A(X,Y) \\ & + \left(2\lambda^2\frac{u_1'}{u_1} - 2u_2\left(2\frac{u_1'}{u_1} - \frac{u_2'}{u_2}\right) + (4-\lambda^2)\frac{u_1'}{u_1} - \lambda u_1\right) \cdot B(X,Y), \end{aligned}$$

concluding the proof. □

Our next goal is to factor the quadratic polynomials $F(X), G(X)$ in a suitable finite extension of the function field $\kappa(\tilde{\mathcal{C}})$; this function field extension will be of the form $\kappa(\tilde{\mathcal{D}})$ for a cover $\tilde{\mathcal{D}} \to \tilde{\mathcal{C}}$. The next Lemma will estimate the Euler characteristic of $\tilde{\mathcal{D}}$.

Lemma 3.9. *Suppose the polynomials $F(X), G(X)$ defined above are both non-constant. There exists a cover $\tilde{\mathcal{D}} \to \tilde{\mathcal{C}}$ of degree ≤ 4 with the following property: let $U \subset \tilde{\mathcal{D}}$ be the set formed by the pre-images of the zeros of the leading and constant coefficients of F and G and the pre-images of S and T. Then the Euler characteristic of $\tilde{\mathcal{D}} \setminus U$ verifies*

$$\chi_U(\tilde{\mathcal{D}}) \leq 30 \cdot \chi_S(\tilde{\mathcal{C}}) + 5 \cdot \max\{0, 2g(\tilde{\mathcal{C}}) - 2\}.$$

Moreover the polynomials $F(X), G(X)$ split in linear factors in the ring $\kappa(\tilde{\mathcal{D}})[X]$.

Proof. Let us define the cover $p : \tilde{\mathcal{D}} \to \tilde{\mathcal{C}}$ (where the curve $\tilde{\mathcal{D}}$ is complete and non-singular) by the property that the field extension $\kappa(\tilde{\mathcal{D}})/p^*(\kappa(\tilde{\mathcal{C}}))$ be the splitting field of $F(X) \cdot G(X)$ over $\kappa(\tilde{\mathcal{C}})$. Clearly the field $\kappa(\tilde{\mathcal{D}})$ is generated over $p^*(\kappa(\tilde{\mathcal{C}}))$ by the square roots of the discriminants of $F(X)$ and $G(X)$, hence has degree ≤ 4, proving our first contention. The cover $p : \tilde{\mathcal{D}} \to \tilde{\mathcal{C}}$ can be ramified only over the zeros and poles of such discriminants, and the ramification index is in any point at most two. The poles are contained in $S \cup T$ and the number of zeros of the discriminants is bounded by their heights. The discriminant of $F(X)$ is

$$(\lambda^2 - 4)\left(\frac{u_2'}{u_2}\right)^2 - 2\lambda^2 \frac{u_2'}{u_2} + 4\frac{u_1'}{u_1}\frac{u_2'}{u_2};$$

we bound its height by estimating its possible poles, which are either poles of u_2'/u_2 or poles of u_1'/u_1. The sum (with multiplicity) is then bounded by $2H_{\tilde{\mathcal{C}}}(u_2'/u_2) + H_{\tilde{\mathcal{C}}}(u_1'/u_1)$. By Lemma 3.5 this quantity

is bounded by $3\chi_S(\tilde{\mathcal{C}})$. The same arguments show that the height of the second discriminant is at most $4H_{\tilde{\mathcal{C}}}(u_1'/u_1)+2H_{\tilde{\mathcal{C}}}(u_2'/u_2)$, which by Lemma 3.5 is bounded by $6\chi_S(\tilde{\mathcal{C}})$. Hence the number of ramification points is at most $\sharp(T\cup S)+9\chi_S(\tilde{\mathcal{C}})$. The Riemann-Hurwitz genus formula gives

$$2g(\tilde{\mathcal{D}})-2=(\deg p)(2g(\tilde{\mathcal{C}})-2)+\sum_{P\in\tilde{\mathcal{D}}}(e_P-1) \tag{3.8}$$

where $e_P\in\{1,2\}$ is the ramification index of the morphism p at the point P. By the above estimate we have

$$\sum_{P\in\tilde{\mathcal{D}}}(e_P-1)\leq \sharp(T\cup S)+9\chi_S(\tilde{\mathcal{C}}). \tag{3.9}$$

Let $U\subset\tilde{\mathcal{D}}$ be the finite set appearing in the statement and let $p(U)\subset\tilde{\mathcal{C}}$ be its image. Then $\sharp U\leq[\kappa(\tilde{\mathcal{D}}):p^*(\kappa(\tilde{\mathcal{C}}))]\cdot\sharp p(U)$; this inequality, combined with the inequality (3.9) and the equality (3.8) gives

$$\begin{aligned}2g(\tilde{\mathcal{D}})-2+\sharp(U)&\leq(\deg p)(2g(\tilde{\mathcal{C}})-2+\sharp p(U))+\sharp(T\cup S)+9\chi_S(\tilde{\mathcal{C}})\\&=(\deg p)(2g(\tilde{\mathcal{C}})-2+\sharp(S\cup T))+(\deg p)\sharp(p(U)\setminus(S\cup T))\\&+\sharp(T\cup S)+9\chi_S(\tilde{\mathcal{C}})\\&\leq 4(2g(\tilde{\mathcal{C}})-2+\sharp(S\cup T))+4\sharp(p(U)\setminus(S\cup T))+\sharp(T\cup S)+9\chi_S(\tilde{\mathcal{C}}),\end{aligned}$$

where we have used the non-negativity of the Euler characteristic $2g(\tilde{\mathcal{C}})-2+\sharp(S\cup T)=\chi_S(\tilde{\mathcal{C}})$. Observe now that the points in $p(U)$ that are not in $S\cup T$ are just the zeros of the leading and constant terms in $F(X)$ and $G(X)$, so they are zeros of u_1'/u_1, u_2'/u_2, $u_1'/u_1-u_2'/u_2$ or of $2u_1'/u_1-u_2'/u_2$; since all of these rational functions are of the form u'/u for suitable S-units u, their heights are bounded by $\chi_S(\tilde{\mathcal{C}})$ (Lemma 3.5); therefore there are at most $4\chi_S(\tilde{\mathcal{C}})$ such points. From the above displayed inequality we then obtain

$$\begin{aligned}\chi_U(\tilde{\mathcal{D}})&\leq 4\chi_{S\cup T}(\tilde{\mathcal{C}})+16\chi_S(\tilde{\mathcal{C}})+\sharp(T)+\sharp(S)+9\chi_S(\tilde{\mathcal{C}})\\&\leq 29\chi_S(\tilde{\mathcal{C}})+5\sharp(T)+\sharp(S)\\&\leq 30\cdot\chi_S(\mathcal{C})+5\max\{0,2g(\tilde{\mathcal{C}})-2\},\end{aligned}$$

finishing the proof. □

We are now able to prove the following

Lemma 3.10. *Let (u_1,u_2,y) be a solution of (3.3) such that $\frac{u_1'}{u_1}\neq\frac{u_2'}{u_2}$, $2\frac{u_1'}{u_1}\neq\frac{u_2'}{u_2}$ and u_1,u_2 are both non constant. Let $\tilde{\mathcal{D}},U$ be as in Lemma 3.9. There exist U-units $a,b\in\kappa(\tilde{\mathcal{D}})$ such that*

$$|\max\{H_{\tilde{\mathcal{D}}}(a),H_{\tilde{\mathcal{D}}}(b)\}-\max\{H_{\tilde{\mathcal{D}}}(u_1),H_{\tilde{\mathcal{D}}}(u_2)\}|\leq 16\cdot\chi_S(\tilde{\mathcal{C}}) \tag{3.12}$$

and

$$\sum_{v\in\tilde{\mathcal{D}}\setminus U}\min\{v(a-1),v(b-1)\}\geq\frac{1}{4}\cdot\sum_{v\in\tilde{\mathcal{D}}\setminus U}v(y). \tag{3.13}$$

Moreover, $a=u_1\alpha^{-1},b=u_2\beta^{-1}$ for suitable α,β with $F(\alpha)=G(\beta)=0$.

In the right-term in (3.13), the rational function $y\in\kappa(\tilde{\mathcal{C}})\subset\kappa(\tilde{\mathcal{D}})$ is viewed as a rational function on $\tilde{\mathcal{D}}$.

Proof. Write the polynomials $F(X),G(X)$ in (3.7) as

$$\begin{aligned}F(X)&=\left(\frac{u_2'}{u_2}-\frac{u_1'}{u_1}\right)(X-\alpha)\cdot(X-\bar{\alpha}),\\G(X)&=2\left(2\frac{u_1'}{u_1}-\frac{u_2'}{u_2}\right)(X-\beta)\cdot(X-\bar{\beta})\end{aligned}$$

where $\alpha, \bar\alpha$ (resp. $\beta, \bar\beta$) are the roots of $F(X)$ (resp. $G(X)$). Recall now that by Lemma 3.8, $F(u_1)$ and $G(u_2)$ are both divisible by y in the ring of U-integers and that the leading and constant coefficients of $F(X)$ and $G(X)$ are U-units, so $\alpha, \bar\alpha, \beta, \bar\beta$ too are U-units. Dividing $F(u_1)$ by the U-unit $\alpha\bar\alpha(\frac{u_2'}{u_2} - \frac{u_1'}{u_1})$ and $G(u_2)$ by the U-unit $\beta\bar\beta 2(2\frac{u_1'}{u_1} - \frac{u_2'}{u_2})$ we obtain that y divides the resulting quotients in the ring of U-integers, so

$$\sum_{v \in \tilde{\mathcal{D}} \setminus U} \min\{v(u_1\alpha^{-1} - 1) + v(u_1\bar\alpha^{-1} - 1), v(u_2\beta^{-1} - 1) + v(u_2\bar\beta^{-1} - 1)\} \geq \sum_{v \in \tilde{\mathcal{D}} \setminus U} v(y).$$

Since

$$\begin{aligned}
&\sum_{v \in \tilde{\mathcal{D}} \setminus U} \min\{v(u_1\alpha^{-1} - 1) + v(u_1\bar\alpha^{-1} - 1), v(u_2\beta^{-1} - 1) + v(u_2\bar\beta^{-1} - 1)\} \\
&\leq \sum_{v \in \tilde{\mathcal{D}} \setminus U} \min\{v(u_1\alpha^{-1} - 1), v(u_2\beta^{-1} - 1) + v(u_2\bar\beta^{-1} - 1)\} \\
&+ \sum_{v \in \tilde{\mathcal{D}} \setminus U} \min\{v(u_2\bar\alpha^{-1} - 1), v(u_2\beta^{-1} - 1) + v(u_2\bar\beta^{-1})\} \\
&\leq \sum_{v \in \tilde{\mathcal{D}} \setminus U} \min\{v(u_1\alpha^{-1} - 1), v(u_2\beta^{-1} - 1)\} + \sum_{v \in \tilde{\mathcal{D}} \setminus U} \min\{v(u_1\alpha^{-1} - 1), v(u_2\bar\beta^{-1} - 1)\} \\
&+ \sum_{v \in \tilde{\mathcal{D}} \setminus U} \min\{v(u_1\bar\alpha^{-1} - 1), v(u_2\beta^{-1} - 1)\} + \sum_{v \in \tilde{\mathcal{D}} \setminus U} \min\{v(u_1\bar\alpha^{-1} - 1), v(u_2\bar\beta^{-1} - 1)\}
\end{aligned}$$

we obtain that there exist U-units $a \in \{u_1\alpha^{-1}, u_2\bar\alpha^{-1}\}$ and $b \in \{u_2\beta^{-1}, u_2\bar\beta^{-1}\}$ such that (3.13) holds. We now estimate the difference in (3.12). Clearly it is bounded by $\max\{H_{\tilde{\mathcal{D}}}(\alpha), H_{\tilde{\mathcal{D}}}(\bar\alpha), H_{\tilde{\mathcal{D}}}(\beta), H_{\tilde{\mathcal{D}}}(\bar\beta)\}$. Since $\alpha, \bar\alpha$ are roots of $F(X)$, which is defined explicitely in (3.7), their poles must be either zeros of $\frac{u_2'}{u_2} - \frac{u_1'}{u_1}$ or be poles of $\frac{u_2'}{u_2}$. Since the $\tilde{\mathcal{C}}$-heights of $\frac{u_2'}{u_2} - \frac{u_1'}{u_1}$ and of $\frac{u_2'}{u_2}$ is bounded by $\chi_S(\tilde{\mathcal{D}})$, their $\tilde{\mathcal{D}}$-height is $\leq 4\chi_S(\tilde{\mathcal{C}})$. Hence

$$\max\{H_{\tilde{\mathcal{D}}}(\alpha), H_{\tilde{\mathcal{D}}}(\bar\alpha)\} \leq 8\chi_S(\tilde{\mathcal{C}}).$$

For the same reason

$$\max\{H_{\tilde{\mathcal{D}}}(\beta), H_{\tilde{\mathcal{D}}}(\bar\beta)\} \leq 16\chi_S(\tilde{\mathcal{C}}). \tag{3.14}$$

□

We now bound from below the height of y in terms of the heights of u_1, u_2 (so also in terms of $H(a), H(b)$). To this end we shall use the following lemma, easily deduced from Theorem 1 in [Z1]:

Lemma 3.11. *Let $\tilde{\mathcal{D}}, U$ be as before, $m \geq 2$ be an integer, $\theta_1, \ldots, \theta_m$ be U-units such that no sum $\sum_{i \in I} \theta_i$ vanishes for any nonempty subset $I \subset \{1, \ldots, m\}$. Then the U-integer $\theta_1 + \ldots + \theta_m$ satisfies*

$$\sum_{v \in \tilde{\mathcal{D}} \setminus U} v(\theta_1 + \ldots + \theta_m)\} \geq H_{\tilde{\mathcal{D}}}(\theta_1 : \ldots : \theta_m) - \binom{m}{2}\chi_U(\tilde{\mathcal{D}}).$$

In particular the height $H_{\tilde{\mathcal{D}}}(\theta_1 + \ldots + \theta_m)$ is bounded from below by the right side term above. □

We shall apply the above Lemma with $m = 4$ to y^2 which is expressed in (3.3) as a sum of four U-units. We obtain, in the case where no subsum in the expression $u_1^2 + \lambda u_1 + u_2 + 1$ vanishes, the following:

Lemma 3.12. *For every solution (y, u_1, u_2) in $\mathcal{O}_U \times (\mathcal{O}_U^*)^2$ to equation (3.3), with no vanishing subsum on the right-side term, we have the lower bound*

$$H_{\tilde{\mathcal{D}}}(y) \geq \sum_{v \in \tilde{\mathcal{D}} \setminus U} v(y) \geq \max\{H_{\tilde{\mathcal{D}}}(u_1), H_{\tilde{\mathcal{D}}}(u_2)\} - 6\chi_U(\tilde{\mathcal{D}}).$$

Proof. It suffices to observe that the projective height $H_{\tilde{\mathcal{D}}}(u_1^2 : \lambda u_1 : u_2 : 1)$ is $\geq \max\{2H_{\tilde{\mathcal{D}}}(u_1), H_{\tilde{\mathcal{D}}}(u_2)\}$. $\square$

¿From inequalities (3.13), (3.12) of Lemma 3.10 and from the above Lemma 3.12 we obtain that for each solution (y, u_1, u_2) of (3.3) there exists a pair of U-units (a, b) such that

$$\sum_{v \in \tilde{\mathcal{D}} \setminus U} \min\{v(a-1), v(b-1)\} \geq \frac{1}{4} \max\{H_{\tilde{\mathcal{D}}}(a), H_{\tilde{\mathcal{D}}}(b)\} - 6\chi_U(\tilde{\mathcal{D}}) - 4\chi_S(\tilde{\mathcal{C}}).$$

Since $\chi_S(\tilde{\mathcal{C}}) \leq \chi_U(\tilde{\mathcal{D}})$ we can deduce from the last inequality

$$\sum_{v \in \tilde{\mathcal{D}} \setminus U} \min\{v(a-1), v(b-1)\} \geq \frac{1}{4} \max\{H_{\tilde{\mathcal{D}}}(a), H_{\tilde{\mathcal{D}}}(b)\} - 10\chi_U(\tilde{\mathcal{D}}). \tag{3.15}$$

We are now able to apply the Corollary 2.3 from §2, to deduce the following

Proposition 3.13. *Let $(y, u_1, u_2) \in \mathcal{O}_S \times (\mathcal{O}_S^\times)^2$ be a solution of equation (3.3) with no vanishing subsum in the right-side term. Suppose also that u_1, u_2 are both non constant and $\frac{u_1'}{u_1} \neq \frac{u_2'}{u_2}$, $2\frac{u_1'}{u_1} \neq \frac{u_2'}{u_2}$. Let $U, \tilde{\mathcal{D}}$ be defined in Lemma 3.9 and a, b be the U-units in $\kappa(\tilde{\mathcal{D}})$ defined in Lemma 3.10. Then either*

$$\max\{H_{\tilde{\mathcal{C}}}(u_1), H_{\tilde{\mathcal{C}}}(u_2)\} \leq 2^{14} \cdot 35 \cdot \chi_S(\tilde{\mathcal{C}}) \tag{3.16}$$

or a, b verify a multiplicative dependence relation of the form

$$a^r \cdot b^s = 1$$

for integers $(r, s) \in \mathbf{Z}^2 \setminus \{0\}$ with

$$\max\{|r|, |s|\} \leq 5. \tag{3.17}$$

Proof. We suppose (3.16) does not hold, and want to prove that a, b verify such a multiplicative dependence relation; since $H_{\tilde{\mathcal{C}}} \leq H_{\tilde{\mathcal{D}}}$, we have a fortiori the lower bound

$$\max\{H_{\tilde{\mathcal{D}}}(u_1), H_{\tilde{\mathcal{D}}}(u_2)\} \geq 2^{14} \cdot 35 \cdot \chi_S(\tilde{\mathcal{C}}) \geq 2^{14}\chi_U(\tilde{\mathcal{D}})$$

the last inequality following easily from Lemma 3.9. By (3.12) we have

$$\max\{H_{\tilde{\mathcal{D}}}(a), H_{\tilde{\mathcal{D}}}(b)\} \geq \max\{H_{\tilde{\mathcal{D}}}(u_1), H_{\tilde{\mathcal{D}}}(u_2)\} - 16\chi_S(\tilde{\mathcal{C}}) \geq \max\{H_{\tilde{\mathcal{D}}}(u_1), H_{\tilde{\mathcal{D}}}(u_2)\} - 16\chi_U(\tilde{\mathcal{D}})$$

so we get from the last two displayed inequalities that

$$\max\{H_{\tilde{\mathcal{D}}}(a), H_{\tilde{\mathcal{D}}}(b)\} \geq (2^{14} - 16)\chi_U(\tilde{\mathcal{D}}). \tag{3.18}$$

Put $H := \max\{H_{\tilde{\mathcal{D}}}(a), H_{\tilde{\mathcal{D}}}(b)\}$ and $\chi := \chi_U(\tilde{\mathcal{D}})$; we check that from (3.15) and (3.18) it follows that

$$\sum_{v \in \tilde{\mathcal{D}} \setminus U} \min\{v(1-a), v(1-b)\} > 3 \cdot 2^{1/3} H^{2/3} \chi^{1/3}. \tag{3.19}$$

In fact, after (3.15) it suffices to show that $\frac{1}{4}H - 10\chi > 3 \cdot 2^{1/3} H^{2/3} \chi^{1/3}$ i.e.

$$H^{2/3}\left(\frac{1}{4}H^{1/3} - 3 \cdot 2^{1/3}\chi^{1/3}\right) > 10\chi.$$

From (3.18) it follows that $\frac{1}{4}H^{1/3} - 3\cdot 2^{1/3}\chi^{1/3} > 3\cdot 2^{1/3}\chi^{1/3}$; also it follows that $H^{2/3} > 3^{-1}\cdot 2^{-1/3}\cdot 10\chi^{2/3}$, concluding the verification of (3.19). Then Corollary 2.3 (i) implies that a, b are multiplicative dependent, i.e. a relation of the form $a^r b^s = 1$ holds, for suitable integers r, s not both zero. Now the part (ii) of the same corollary gives the bound

$$\sum_{v\in\tilde{\mathcal{D}}\setminus U} \min\{v(a-1), v(b-1)\} \leq \frac{\max\{H_{\tilde{\mathcal{D}}}(a), H_{\tilde{\mathcal{D}}}(b)\}}{\max\{|r|,|s|\}}.$$

The above bound and the inequality (3.15) and (3.18) give, with the previous notation for H, χ,

$$\frac{H}{\max\{|r|,|s|\}} > \frac{1}{4}H - 10\chi > \frac{1}{5}H,$$

so $\max\{|r|,|s|\} \leq 5$. □

Proposition 3.13 does not give the full conclusion of Theorem 3.4, since it provides a multiplicative dependence relation between a and b, not u_1, u_2. To end the proof of Theorem 3.4, we need a last lemma:

Lemma 3.14. *Let $\tilde{\mathcal{D}}$ be as before; let $u_1, u_2 \in \kappa(\tilde{\mathcal{D}})$ be non zero rational functions. Let $A(X,Y) \in \kappa[X,Y]$ be a polynomial and let $B(X,Y) \in \kappa(\tilde{\mathcal{D}})[X,Y]$ be defined, in terms of $A(X,Y)$, u_1, u_2 as*

$$B(X,Y) = \frac{u_1'}{u_1}X\frac{\partial}{\partial X}A(X,Y) + \frac{u_2'}{u_2}Y\frac{\partial}{\partial Y}A(X,Y).$$

Let $F(X) \in \kappa(\tilde{\mathcal{C}})[X]$, $G(Y) \in \kappa(\tilde{\mathcal{D}})[Y]$ be the resultants of $A(X,Y), B(X,Y)$ with respect to Y and X respectively. Finally let $\alpha \in \kappa(\tilde{\mathcal{D}})$ be a root of $F(X)$ and $\beta \in \kappa(\tilde{\mathcal{D}})$ a root of $G(Y)$. If $u_1\alpha^{-1}$ and $u_2\beta^{-1}$ satisfy a multiplicative dependence relation of the form

$$\left(\frac{u_1}{\alpha}\right)^r \cdot \left(\frac{u_2}{\beta}\right)^s = \mu, \tag{3.20}$$

for a constant $\mu \in \kappa$, then either one between u_1/α and u_2/β is constant or u_1, u_2 satisfy a multiplicative dependence relation of the same type, i.e. $u_1^r \cdot u_2^s \in \kappa$.

Proof. If u_1 or u_2 are constant, the conclusion follows easily, so we suppose them to be non-constant. We first observe that multiplicative dependence relations of the above type correspond to linear relations over the rationals of the form

$$r\left(\frac{u_1'}{u_1} - \frac{\alpha'}{\alpha}\right) + s\left(\frac{u_2'}{u_2} - \frac{\beta'}{\beta}\right) = 0. \tag{3.21}$$

We want to prove that the corresponding linear relation $r\frac{u_1'}{u_1} + s\frac{u_2'}{u_2} = 0$ holds for u_1, u_2, unless the factor of r or s in the left term of (3.21) vanishes. Starting from the relation $A(\alpha,\beta) = 0$ we obtain, by taking differentials,

$$\alpha'\frac{\partial}{\partial X}A(\alpha,\beta) + \beta'\frac{\partial}{\partial Y}A(\alpha,\beta) = 0.$$

Since $B(X,Y)$ too vanishes at the point (α,β) we get, applying the definition of $B(X,Y)$,

$$\frac{u_1'}{u_1}\alpha\frac{\partial}{\partial X}A(\alpha,\beta) + \frac{u_2'}{u_2}\beta\frac{\partial}{\partial Y}A(\alpha,\beta) = 0.$$

¿From the two above relations we obtain that

$$\frac{u_1'}{u_1}\cdot\left(\frac{u_2'}{u_2}\right)^{-1} = \frac{\alpha'/\alpha}{\beta'/\beta}.$$

This implies that the matrix $\begin{pmatrix} \frac{u_1'}{u_1} & \frac{u_2'}{u_2} \\ \frac{\alpha'}{\alpha} & \frac{\beta'}{\beta} \end{pmatrix}$ has rank one, so the same holds for the matrix

$$\begin{pmatrix} \frac{u_1'}{u_1} - \frac{\alpha'}{\alpha} & \frac{u_2'}{u_2} - \frac{\beta'}{\beta} \\ \frac{u_1'}{u_1} & \frac{u_2'}{u_2} \end{pmatrix}.$$

If the first row in the above matrix vanishes we are done, since then u_1/α (and u_2/β) would be constant. Then we shall suppose that the first row does not vanishes; since it is orthogonal to the vector (r, s) by (3.21), and the matrix has rank one, the same must hold for the second row, obtaining the sought conclusion that $r\frac{u_1'}{u_1} + s\frac{u_2'}{u_2} = 0$. □

Proof of Theorem 3.4. We shall suppose that conditions (i), (ii) and (iii) of Theorem 3.4 are not satisfied and deduce a contradiction. Since (i) is not satisfied no subsum in the right term of (3.3) can vanish. Also, the polynomial $F(X)$ defined by (3.7) has degree exactely two, in particular is not constant, because the vanishing of its leading coefficient (which is $\frac{u_1'}{u_1} - \frac{u_2'}{u_2}$) would imply a multiplicative relation of the form $u_1 = \mu \cdot u_2$, for a constant $\mu \in \kappa^*$, which has been excluded by (ii). The same remarks applies to the leading coefficient of $G(X)$, which vanishes only if $u_1 \cdot u_2^{-2}$ is a constant. Also $\frac{u_1'}{u_1} \neq 0$ and $\frac{u_2'}{u_2} \neq 0$, otherwise u_1 or u_2 would be constant, hence the pair (u_1, u_2) would satisfy a multiplicative dependence relation with exponents $(1, 0)$ or $(0, 1)$. Then neither $F(X)$ nor $G(X)$ are constant polynomials and their constant coefficient are nonzero.

Then the above lemmas apply and we obtain that for a suitable root α, (resp. β) of the polynomial $F(X)$ (resp. $G(X)$) defined in (3.7), the U-units $a = u_1\alpha^{-1}$ and $b = u_2\beta^{-1}$ satisfy a multiplicative dependence relation as in Proposition 3.13. But then, by the above Lemma, either u_1/α is constant, or u_2/β is constant or u_1, u_2 too satisfy a multiplicative dependence relation (with the same exponents as for a and b). The first two possibilities are ruled out by height consideration: in fact if u_1/α were constant, the height of u_1 would be equal to the height of α which is bounded by the inequality (3.14); since we assume that (iii) of Theorem 3.4 does not hold, this is excluded. For the same reason u_2/β is not constant. The last possibility coincides with (ii), which we have excluded, concluding the proof. □

We now prove Theorem 3.1. which, as we already remarked, is equivalent to Theorem 1.1.

Proof of Theorem 3.1. By Lemma 3.3 we can find a scalar $\lambda \in \kappa$ and identify X with the open subset of the plane $\mathbf{A}^2$ defined by

$$x \cdot (y^2 - x^2 - \lambda x - 1) \neq 0.$$

Then, as in Lemma 3.3, the morphism $f : \tilde{\mathcal{C}} \setminus S \to X$ will be expressed in the form

$$f : p \mapsto (u_1(p), y(p))$$

where $(y, u_1) \in \mathcal{O}_S \times (\mathcal{O}_S)^*$ satisfies (3.3) for a suitable S-unit $u_2 \in \mathcal{O}_S^*$. Then one of the conclusions (i),...,(iii) of Theorem 3.4 holds. We shall prove the estimate of Theorem 3.1 in each case:

- (i) suppose that some subsum on the right term of (3.3) vanishes; then the curve $f(\tilde{\mathcal{C}})$ is either a line or a conic, hence its degree is ≤ 2.
- (ii) if $u_2^s = u_1^r \cdot \mu$, where $\mu \in \kappa$ and the pair of integers $(r, s) \in \mathbf{Z}^2 \setminus \{0\}$ satsify $\max\{|r|, |s|\} \leq 5$, then the degree of $f(\tilde{\mathcal{C}})$ is ≤ 20;
- (iii) finally, if $\max\{H_{\tilde{\mathcal{C}}}(u_1), H_{\tilde{\mathcal{C}}}(u_2)\} \leq 2^{20} \cdot 35 \cdot \chi(\mathcal{C})$, then by Lemma 3.2 the estimate of Theorem 3.1 holds. □

Proof of Theorem 1.2. We now give a sketch of the proof of the full Theorem 1.2, of which Theorem 3.4 (hence also Theorems 1.1 and 3.1) is just a particular cases. We avoid however the rather lengthy explicit calculations of the constants involved in the estimates.

Suppose $A(u_1,u_2)$ has "many" multiple zeros, with S-units $u_1,u_2\in\mathcal{O}_S$. Taking the differential of $A(u_1,u_2)$, and using again the above formalism, we obtain a second function $B(u_1,u_2)$ with many zeros in common with $A(u_1,u_2)$, where the polynomial $B(X,Y)\in\kappa(\tilde{\mathcal{C}})[X,Y]$ is defined in the statement of Lemma 3.7. The height of its coefficients can be easily estimated in terms of $\chi_S(\tilde{\mathcal{C}})$. Taking again the resultants $F(X),G(Y)$ we obtain an analogue of Lemma 3.8, so a "large" gcd between $F(u_1)$ and $G(u_2)$. After factoring $F(X)$ and $G(Y)$ in linear factors in a suitable field field extension of $\kappa(\tilde{\mathcal{C}})$, we reduce to the same situation as in the above proof, and conclude in the same way. We note that Lemma 3.14 still applies in this context.

Proof of Theorem 1.3. It can be easily deduced from Theorem 1.2. In fact the function field of Y is a cyclic extension of the function field of $\mathbf{G}_m^2$, so it is obtained by taking the d-th root of a suitable rational function on $\mathbf{G}_m^2$. Since by hypothesis there exists a finite map $\pi:Y\to\mathbf{G}_m^2$ we can write the κ-algebra $\kappa[Y]$ as the extension of the κ-algebra $\kappa[\mathbf{G}_m^2]=\kappa[u_1,u_2]$ obtained by adding a d-th root:

$$\kappa[Y]=\kappa[u_1,u_2,\frac{1}{u_1},\frac{1}{u_2}][\sqrt[d]{F(u_1,u_2)}]$$

for a suitable polynomial $F(X,Y)\in\kappa[X,Y]$ which is not a perfect d-th power. Let now $\tilde{\mathcal{C}}$ be a curve, $S\subset\tilde{\mathcal{C}}$ a finite set as before and $f:\tilde{\mathcal{C}}\setminus S\to Y$ a morphism. Composing it with π we obtain a morphism $\pi\circ f:\tilde{\mathcal{C}}\setminus S\to\mathbf{G}_m^2$ which is expressed as $(\tilde{\mathcal{C}}\setminus S)\ni p\mapsto(u_1(p),u_2(p))\in\mathbf{G}_m^2$ for suitable S-units u_1,u_2 in $\kappa(\tilde{\mathcal{C}})$. The fact that such a morphism factors through Y just means that the rational function $F(u_1,u_2)$ is a perfect d-th power. Then we can apply Theorem 1.2 and deduce either a bound on the height of such a morphism or a multiplicative dependence relation between u_1 and u_2. In both cases one obtains a bound for the projective degree of the image $\pi\circ f(\tilde{\mathcal{C}})$ in $\mathbf{G}_m^2\subset\mathbf{P}_2$, which in turn gives a bound for the projective degree of the curve $f(\tilde{\mathcal{C}})$ with respect to the given embedding.

§4. Maps to the complement of a cubic.

Since the canonical divisor of $\mathbf{P}_2$ has degree -3, the complement of a smooth cubic $D\subset\mathbf{P}_2$, or of a singular cubic with normal crossing singularities, has vanishing log Kodaira dimension. Hence one expects that the conclusion of Theorem 1.1 does not hold in this case, so the degree of a curve in $\mathbf{P}_2\setminus D$ cannot be bounded in term of its Euler characteristic. We prove that this is the case for reducible cubics: this is the content of Theorem 1.4 which we now prove.

We distinguish two cases, according to the number of components of D; we shall prove separately the following assertions, which will immediately imply Theorem 1.4:

Proposition 4.1. *Let $D\subset\mathbf{P}_2$ be the union of a smooth conic and a secant line and let P,Q be the singular points of D. For each positive number n there exists a morphism $f:\mathbf{P}_1\to\mathbf{P}_2$ such that the image $\mathcal{C}:=f(\mathbf{P}_1\setminus\{0,\infty\})$ is contained in $\mathbf{P}_2\setminus D$ and the projective curve $f(\mathbf{P}_1)$ has degree n. In particular the affine curve $\mathcal{C}\subset\mathbf{P}_2\setminus D$ verifies* $\deg(\mathcal{C})=n$ *and* $\chi(\mathcal{C})=0$.

Proof. In a suitable system of homogeneous coordinates $(x_0:x_1:x_2)$, the conic will have the equation $x_1^2-x_2^2=x_0^2$ and the line will be defined by $x_0=0$. Putting $x:=x_1/x_0$, $y=x_2/x_0$, $\mathbf{P}_2\setminus D$ will be the complement in $\mathbf{A}^2$ of the hyperbola of equation $x^2-y^2=1$ whose points at infinity P and Q will have coordinates $P=(0:1:-1)$ and $(0:1:1)$. Consider, for $n>1$, the morphism $f:\mathbf{G}_m\to\mathbf{A}^2$ defined by

$$f(t)=\left(\frac{t^2-t^n+1}{2t},\frac{1-t^n-t^2}{2t}\right)=(x(t),y(t))$$

and observe that $x(t)^2-y(t)^2=1-t^n$. Then for no $t\in\mathbf{G}_m$ the point $f(t)$ can belong to the curve of equation $x^2-y^2=1$. On the contrary, the continuation of f to a morphism $\mathbf{P}_1\to\mathbf{P}_2$ sends 0 to the point at infinity P and ∞ to the point at Q. We have then obtained a curve $\mathcal{C}=f(\mathbf{G}_m)\subset\mathbf{P}_2\setminus D$ of zero Euler characteristic. We now calculate its degree. Putting for simplification $\xi:=(x+y)$, $\eta:=x-y$ one obtains $\xi=(1-t^n)/t$ and $\eta=t$, so ξ, η satisfy the irreducible equation of degree n

$$\xi\eta=1-\eta^n.$$

$\square$

Proposition 4.2. *Let $D \subset \mathbf{P}_2$ be the union of three lines in general position. Then the complement $\mathbf{P}_2 \setminus D$ is isomorphic to the torus $\mathbf{G}_m^2$. For each positive n there exists a one-dimensional subtorus $\mathcal{C} \subset \mathbf{G}_m^2 \simeq \mathbf{P}_2 \setminus D$, which has degree n as a curve in $\mathbf{P}_2$; its Euler characteristic is 0.*

Proof. Choosing suitable homogeneous coordinates, the divisor D has equation $x_0x_1x_2 = 0$. The morphism $\mathbf{G}_m \ni t \mapsto (1 : t : t^n)$ maps $\mathbf{G}_m$ to $\mathbf{P}_2 \setminus D$. The image is the degree n curve of equation $x_1^n = x_2$; it is also a one-dimensional algebraic subgroup with respect to the canonical algebraic group structure of $\mathbf{P}_2 \setminus D \simeq \mathbf{G}_m^2$. $\square$

We end this paragraph by showing that the condition on the normal crossing singularities of D cannot be removed. In fact we have

Proposition 4.3. *Let D be the sum of a smooth conic and two lines meeting on the conic. Let $X = \mathbf{P}_2 \setminus D$. For every positive n there exists a curve of degree n and vanishing Euler characteristic on X.*

We can choose affine coordinates x, y for the complement of the first line on $\mathbf{P}_2$, identified with $\mathbf{A}^2$, so that the second line has equation $x = 0$ and the conic $(x-1)y + 1 = 0$. Then the image of the morphism $f : \mathbf{G}_m \to \mathbf{P}_2 \setminus D$ defined by

$$f(t) = \left(t, \frac{t^{n+1}-1}{t-1}\right) = (x(t), y(t))$$

avoids both the line of equation $x = 0$ and the conic of equation $(x-1)y+1 = 0$. Clearly it is a plane curve of degree n. $\square$

Appendix

The upper bound of Corollary 2.3 for the number of common zeros outside S of $a-1, b-1$ is related to the paper [BMZ1], especially to Thm. 2 therein, which in a different but equivalent language states: *Let $\mathcal{C} \subset \mathbf{G}_m^n$ be an irreducible curve over $\overline{\mathbf{Q}}$. Suppose that the coordinate functions $x_1, \ldots, x_n$ on $\mathcal{C}$ are multiplicatively independent modulo constants. Then there are only finitely many points $P \in \tilde{\mathcal{C}}$ such that there are two independent relations $x_1(P)^{a_1} \cdots x_n(P)^{a_n} = x_1(P)^{b_1} \cdots x_n(P)^{b_n} = 1$, $a_i, b_i \in \mathbf{Z}$.* (See [BMZ2] for the case over $\mathbf{C}$.)

On setting $a = x_1^{a_1} \cdots x_n^{a_n}$, $b = x_1^{b_1} \cdots x_n^{b_n}$, we see that each relevant point P in this statement is a common zero of $a-1, b-1$, clarifying the connection with the present context.

For instance Corollary 2.3 (for $\kappa = \overline{\mathbf{Q}}$, $S =$ set of zeros/poles of the x_i) bounds the number of relevant points P, for a *given* choice of the exponents a_i, b_i. In a way, [BMZ1, Thm. 2] is much stronger, in that it asserts the finiteness of the points for *varying* pairs of exponents vectors, leading for instance to a bound $\sum_{v \notin S} \min(v(1-a), v(1-b)) = O_{\tilde{\mathcal{C}},S}(1)$, independently of a, b of the above shape. However the present corollary is rather more explicit and uniform in the dependence with respect to $\tilde{\mathcal{C}}$ and S. This may be crucial for some applications; for instance, the result of [BMZ1] would not be sufficient to derive the present Theorems 1.1 and 1.3; in fact, in the present proofs Corollary 2.3 is applied with a set S (and also a curve $\mathcal{C}$) which vary with the individual solutions, and hence a good uniformity is needed.

Also, Corollary 2.3 would lead to a simplification of the proofs in [BMZ1]; we briefly indicate how, referring to that paper for this argument. We let a, b be as above. Then Corollary 2.3 gives $\sum_{v \notin S} \min(v(1-a), v(1-b)) \ll (H(a)H(b))^{1/3}$.

Say now that $\mathcal{C}$ is defined over the number field L. Then if $a(P) - 1 = b(P) - 1 = 0$ for a $P \in \mathcal{C}$, the same is true for any conjugate of P over L. Then Corollary 2.3 implies at once $[L(P) : L] \ll (H(a)H(b))^{1/3}$. We may also assume that the exponent vectors $\mathbf{a}, \mathbf{b}$ are the first two successive minima for the lattice of exponents in the relations among the $x_i(P)$, with volume V, say. We then have $H(a)H(b) \ll |\mathbf{a}| \cdot |\mathbf{b}| \ll V$. Hence $[L(P) : L] \ll V^{1/3}$. Now, in the notation of [BMZ1], we may take $V \ll N\Pi$, and then the above improves on [BMZ1, (4.4)], in the crucial case $r = n-2$; the gain suffices to avoid the final, somewhat involved, arguments in that proof.

We should mention here also the result by Ailon and Rudnick [AR]: they consider the $\gcd(f^n-1, g^n-1)$ for given polynomials f, g (or more generally for given rational functions on a curve). Using the well-known finiteness of torsion points on curves in $\mathbf{G}_m^n$ not contained in any translate of a proper algebraic subgroups, they show that the gcd is "usually" 1, for multiplicatively independent f, g.

As in the previous situation concerning [BMZ1], a very special case of Corollary 2.3 bounds the gcd-degree for given n, which is on the one hand weaker than [AR]. However, somewhat surprisingly, this bound allows to recover easily the arithmetical result about torsion points. In fact, let x, y be multiplicatively independent rational functions on the curve $\mathcal{C}/L$ and let P be a torsion point, so $x(P)^N = y(P)^N = 1$ for some integer $N \geq 1$ which we suppose to be minimal. Then every conjugate of P over L is a common zero of $x^N - 1$ and $y^N - 1$. Applying Corollary 2.3 we deduce that $[L(P) : L] \ll N^{2/3}$. But since N is minimal, the relevant degree is $\gg \phi(N) \gg N/\log\log N$, proving the boundedness of N.

The argument actually leads to quite explicit bounds (in terms only of the degree and genus of $\mathcal{C}$ and of $[L : \mathbf{Q}]$) whose calculation we shall perform in a future note. Here we only remark that they lead to:

(i) recover the best possible exponent (i.e. 2) for the degree and

(ii) better results than the known ones when the genus is small.

To our knowledge no known estimate involves a dependence on the genus.

It seems not a very common fact that function-field arguments like the present ones, involving derivations, lead directly to arithmetical deductions; it would be interesting to develop function field techniques for similar questions concerning abelian varieties; apart from some independent interest, this could perhaps lead to analogous, much deeper, arithmetical consequences (first obtained in this direction by Raynaud).

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